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Indefinite integration of oscillatory functions by the Chebyshev series expansion

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Abstract: An automatic quadrature scheme is presented for evaluating the indefinite integral of oscillatory function $\int_0^x f(t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t$, $0 \leqslant x \leqslant 1$, of a given function $f(t)$, which is usually assumed to be smooth. The function $f(t)$ is expanded in the Chebyshev series to make an efficient evaluation of the indefinite integral. Combining the automatic quadrature method obtained and Sidi's extrapolation method makes an effective quadrature scheme for oscillatory infinite integral $\int_a^\infty f(x) \cos \omega x\,\mathrm{d}x$ for which numerical examples are also presented.

Keywords: Indefinite integration of oscillatory function, automatic quadrature, Chebyshev series expansion, three term recurrence, oscillatory infinite integral, Sidi's extrapolation.

1. Introduction

We will present an automatic quadrature scheme for the indefinite integral of oscillatory function

$$I(\omega, x) = \int_0^x f(t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\,, \quad 0 \leqslant x \leqslant 1\,, \tag{1}$$

where $f(t)$ is a given function, usually assumed to be smooth. There seems to exist no automatic quadrature scheme for evaluating $I(\omega, x)$, (1), although a considerable literature has been published for the numerical evaluations of the definite integral of oscillatory function $I(\omega, x=1)$ and the indefinite integral of non-oscillatory function $I(\omega=0, x)$.

For the quadrature scheme for evaluating the definite integral $I(\omega, x=1)$ see, for example [2,7]. On the other hand, the indefinite integral $I(\omega=0, x)$ is easily computed by expanding $f(t)$ in the Chebyshev series

$$f(t) = \sum_{k=0}^{\infty}{}' \, a_k T_k^*(t)\,, \tag{2}$$

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and by integrating the series (2) term by term, see Clenshaw and Curtis [1] and [2]. Here $T_k^*(t)$ is the shifted Chebyshev polynomial of the first kind of degree k given by $T_k^*(t) = T_k(2t-1)$, $T_k(u) = \cos k\theta$ $(u = \cos\theta)$. The prime in (2) denotes the summation whose first term is halved. Filippi [3] proposed a modification to the Clenshaw–Curtis method in which the function $f(t)$ is expanded in terms of the Chebyshev polynomials of the second kind instead of the first kind. If $f(t)$ is sufficiently smooth, then the Chebyshev series (2) is rapidly convergent. Luke [6] gives algorithms for computing the indefinite integral (1) by using the Chebyshev expansion in particular when $f(t)$ is a hypergeometric function. We extend the method due to Luke to an automatic integration for the integral (1) of a given function $f(t)$.

Our method is based on the Chebyshev series expansion (2) of $f(t)$. In practice $f(t)$ is approximated by a finite sum of Chebyshev polynomials

$$p_N(t) = \sum_{k=0}^{N}{}'' a_k^N T_k^*(t), \tag{3}$$

where double prime denotes the summation whose first and last terms are halved. For the method to generate the sequence $\{p_N(t)\}$ recursively, see [12] where the integer N is allowed to take the form 1.5×2^n as well as 2^n, which is usually the case, and the fast cosine transform whose algorithm is based on the FFT is used to efficiently compute the coefficients a_k^N's in (3). We note that our quadrature scheme is of closed type in the sense that the set of the sample points for the interpolation polynomial (3) includes both ends of the interval [0, 1], while in the Filippi method [3] the endpoints are omitted. It is convenient to write the indefinite integral (1) in the form

$$\int_0^x f(t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t = \mathrm{e}^{\mathrm{i}\omega x} g(x) - g(0), \tag{4}$$

because if $f(t)$ is a polynomial of degree N, $g(x)$ is also a polynomial of the same degree N. The function $g(x)$ is also expressed by the series,

$$g(x) = \sum_{k=0}^{\infty}{}' c_k T_k^*(x). \tag{5}$$

In Section 2 we give an algorithm for computing the coefficients c_k's of (5) by using the recurrence formula

$$c_{k-1} + \frac{4k}{\mathrm{i}\omega} c_k - c_{k+1} = \frac{1}{\mathrm{i}\omega}(a_{k-1} - a_{k+1}), \quad 1 \leqslant k. \tag{6}$$

The coefficients c_k's of the Chebyshev series (5) are obtained as minimal solutions [4] of the inhomogeneous recurrence relation (6). The error estimate of the approximation $\int_0^x \mathrm{e}^{\mathrm{i}\omega t} p_N(t)\, \mathrm{d}t$ to the integral (1) is given in Section 3. In Section 4 we apply the quadrature method obtained for the indefinite integral (1) to the integration of the oscillatory infinite integral

$$I^c = \int_a^\infty h(x) \cos\omega x\, \mathrm{d}x \quad \left(\text{or} \int_a^\infty h(x) \sin\omega x\, \mathrm{d}x\right), \tag{7}$$

when we also make use of an extrapolation method due to Sidi [9, 10], which seems to be most suitable for accelerating the convergence of the infinite integral (7). Numerical examples are given in Section 5.

2. Algorithm for computing the Chebyshev coefficients

First, we will derive the recurrence relation (6). Differentiating both the sides of (4) with respect to x yields the first order differential equation for $g(x)$

$$g'(x) + \mathrm{i}\omega g(x) = f(x) . \tag{8}$$

If we use the series (2) and (5) in the integrated form of (8)

$$g(x) - g(0) + \mathrm{i}\omega \int_0^x g(t)\,\mathrm{d}t = \int_0^x f(t)\,\mathrm{d}t , \tag{9}$$

and integrate them term by term, we can easily obtain the recurrence formula (6) by comparing the coefficients of the resulting Chebyshev series.

Now, we show the method for computing the Chebyshev coefficients c_k's of $g(x)$ in (5) by using the recurrence relation (6). Actually, it is sufficient to consider the case that a given function $f(t)$ in (4) is approximated by a polynomial $p_N(t)$ of degree N given by (3). The corresponding function $g(x)$ in (4) is a polynomial of the same degree N. Then it suffices to compute $N+1$ Chebyshev coefficients c_k's $(0 \leqslant k \leqslant N)$ of $g(x)$, which are minimal solutions of the recurrence relation (6) with a_k replaced by coefficients a_k^N's of the finite Chebyshev expansion (3). It is well known that computing the minimal solutions of recurrence relations involves the problem of numerical instability [4].

For $N \leqslant |\omega|/2$, the computation of the minimal solutions by the inhomogeneous recurrence formula (6) in the backward direction is stable. Set $c_{N+1} = c_{N+2} = 0$, then use (6) to compute $c_N, c_{N-1}, \ldots, c_0$. For $N > |\omega|/2$, the backward computation is unstable. This instability can be avoided if the coefficients c_k's are determined by solving a system of linear equations. To this end we define a function $G(x)$, which is also expanded in the infinite Chebyshev series,

$$G(x) = g(x) - \mathrm{e}^{-\mathrm{i}\omega x} g(0) = \sum_{k=0}^{\infty}{}' \, b_k T_k^*(x) . \tag{10}$$

Then the indefinite integral (4) is rewritten in the form

$$\int_0^x f(t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \mathrm{e}^{\mathrm{i}\omega x} G(x) , \quad G(0) = 0 . \tag{11}$$

Using the identity involving the modified Bessel function $I_k(-\mathrm{i}\omega/2)$

$$\mathrm{e}^{-\mathrm{i}\omega x} = 2\,\mathrm{e}^{-\mathrm{i}\omega/2} \sum_{k=0}^{\infty}{}' \, I_k(-\mathrm{i}\omega/2) T_k^*(x) , \tag{12}$$

in (10) and noting that $g(x)$ is a polynomial of degree N we have the Chebyshev coefficients b_k's of $G(x)$

$$b_k = -2\,\mathrm{e}^{-\mathrm{i}\omega/2} g(0) I_k , \quad N+1 \leqslant k , \tag{13}$$

which is written in the alternative form

$$b_k = I_k b_{N+1} / I_{N+1} , \quad N+2 \leqslant k . \tag{14}$$

In (13) and (14) and hereafter we abbreviate $I_k(-\mathrm{i}\omega/2)$ to I_k for simplicity. Substituting (14) into (10) we have the expression for $G(x)$

$$G(x) = \sum_{k=0}^{N+1}{}' \, b_k T_k^*(x) + \left\{ \sum_{k=N+2}^{\infty} I_k T_k^*(x) \right\} b_{N+1}/I_{N+1} \,. \tag{15}$$

Therefore it is required to compute only $N+2$ coefficients b_k's ($0 \leqslant k \leqslant N+1$) in (15) which are also the minimal solutions of (6) with c_k's replaced by b_k's.

Let γ_k denote a particular solution of the inhomogeneous recurrence relation (6), then the coefficients b_k's can be expressed in the form

$$b_k = \gamma_k + \lambda I_k \,, \quad 0 \leqslant k \leqslant N+1 \,, \tag{16}$$

because the modified Bessel function I_k is the minimal solution of the homogeneous recurrence relation (6) when the right hand side of (6) is put to zero. The constant λ in (16) is determined so that b_k's satisfy the condition that $G(0) = 0$, which is written from (15) as follows

$$\sum_{k=0}^{N+1}{}' \, (-1)^k b_k + \left\{ \sum_{k=N+2}^{\infty} (-1)^k I_k \right\} b_{N+1}/I_{N+1} = 0 \,. \tag{17}$$

Specifically, by using identity (12), (16) and (17) we have the constant λ

$$\lambda = -2\,\mathrm{e}^{-\mathrm{i}\omega/2} \left[\sum_{k=0}^{N+1}{}' (-1)^k \gamma_k + \left\{ \sum_{k=N+2}^{\infty} (-1)^k I_k \right\} \gamma_{N+1}/I_{N+1} \right] . \tag{18}$$

After all we need to compute the $N+2$ particular solutions γ_k's ($0 \leqslant k \leqslant N+1$) of (6).

Let $m = [|\omega|/2]$ and set $\gamma_m = 0$, then we may compute the particular solutions γ_k's ($m+1 \leqslant k \leqslant N+1$) by solving the following system of linear equations by LU decomposition

$$\begin{pmatrix} \alpha(m+1) & -1 & & & 0 \\ 1 & \alpha(m+2) & -1 & & \\ & \cdots & \ddots & \ddots & \\ & & 1 & \alpha N & -1 \\ 0 & & & 1 & \beta \end{pmatrix} \begin{pmatrix} \gamma_{m+1} \\ \vdots \\ \vdots \\ \gamma_{N+1} \end{pmatrix} = \begin{pmatrix} d_{m+1} \\ \vdots \\ \vdots \\ d_{N+1} \end{pmatrix}, \tag{19}$$

which is derived from (6) with c_k replaced by γ_k, (14) and (16). For simplicity in (19) we have set $\alpha = 4/(\mathrm{i}\omega)$, $\beta = \alpha(N+1) - I_{N+2}/I_{N+1}$, $d_k = (a_{k-1}^N - a_{k+1}^N)/(\mathrm{i}\omega)$ $(m+1 \leqslant k \leqslant N-1)$, $d_N = a_{N-1}^N/(\mathrm{i}\omega)$, $d_{N+1} = a_N^N/(\mathrm{i}\omega)$. It should be noted that the coefficient matrix in (19) is diagonally dominant. For the computation of γ_k's ($0 \leqslant k \leqslant m$) we make use of the recurrence relation (6) in the backward direction with starting values $\gamma_m = 0$ and γ_{m+1} computed from the solution of (19).

To actually evaluate the indefinite integral (1) for $f(t)$ approximated by a polynomial $p_N(t)$ given by (3) the particular solutions γ_k's ($0 \leqslant k \leqslant N+1$) computed above are used in the right hand side of the following relation

$$\int_0^x p_N(t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \mathrm{e}^{\mathrm{i}\omega x} \sum_{k=0}^{N}{}' \, (\gamma_k - I_k \gamma_{N+1}/I_{N+1}) T_k^*(x) - \sum_{k=0}^{N}{}' \, (-1)^k (\gamma_k - I_k \gamma_{N+1}/I_{N+1}) \,, \tag{20}$$

which is derived from (11), (12), (15), (16) and (18). In the computation of the above right hand side, however, there is a considerable loss of significant digits when $|\gamma_{N+1}| > |I_{N+1}|$. In

set $\alpha = 4/(i\omega)$, $\beta = \alpha(N+1) - I_{N+2}/I_{N+1}$, $d_k = (a_{k-1}^N - a_{k+1}^N)/(\mathrm{i}\omega)$, $(m+1 \leqslant k \leqslant N-1)$, this case we make use of the identity (12) to rewrite (20) as follows

$$\int_0^x p_N(t)\,e^{i\omega t}\,dt = e^{i\omega x}\Big[\sum_{k=0}^{N+1}{}' \gamma_k T_k^*(x) + \Big\{\sum_{k=N+2}^{\infty} I_k T_k^*(x)\Big\}\gamma_{N+1}/I_{N+1}\Big] - \Big[\sum_{k=0}^{N+1}{}' (-1)^k\gamma_k + \Big\{\sum_{k=N+2}^{\infty} (-1)^k I_k\Big\}\gamma_{N+1}/I_{N+1}\Big]. \tag{21}$$

In implementing this quadrature scheme on a computer it is sufficient to approximate the infinite summation $(\gamma_{N+1}/I_{N+1})\,\Sigma_{k=N+2}^{\infty} I_k T_k^*(x)$ in (21) by a finite sum $(\gamma_{N+1}/I_{N+1})\times \Sigma_{k=N+2}^{M} I_k T_k^*(x)$ such that the resulting error satisfies

$$|\gamma_{N+1}/I_{N+1}|\Big|\sum_{k=M+1}^{\infty} I_k T_k^*(x)\Big| \leq |\gamma_{N+1}/I_{N+1}| \sum_{k=M+1}^{\infty} |I_k| = o(\varepsilon)\,, \tag{22}$$

with the tolerance ε for the integral (1). The fact that $|I_k|$ rapidly decreases like $|I_k| \sim (|\omega|/4)^k/k!$ as k increases enables us to estimate $\Sigma_{k=M+1}^{\infty}|I_k|$ in (22) as follows

$$\sum_{k=M+1}^{\infty} |I_k| \simeq \sum_{k=1}^{\infty} (|\omega|/4)^{M+k}/(M+k)!$$
$$< (|\omega|/4)^{M+1}/(M+1)! \sum_{k=0}^{\infty} \{(|\omega|/4)/(M+2)\}^k < 2(|\omega|/4)^{M+1}/(M+1)!\,,$$

where we have used the relation, $|\omega|/2 < M+2$, in the last inequality.

3. Error estimation

We will discuss the error estimate for the approximation $\int_0^x e^{i\omega t} p_N(t)\,dt$ to the integral (1), where the integer N is allowed to take the form 2^n and 1.5×2^n ($n = 1, 2, \ldots$). First assume that $N = 2^n$ ($n = 1, 2, \ldots$). Then the polynomial $p_N(t)$ is determined so as to interpolate a function $f(t)$ at the abscissae t_m's, $0 \leq m \leq N$, which are zeros of $T_{N+1}^*(t) - T_{N-1}^*(t)$ and agree with those used by Clenshaw and Curtis [1] over $[-1, 1]$ if we use $T_k(t)$ instead of $T_k^*(t)$. For details see [5].

Specifically, the interpolation condition for $p_N(t)$ is

$$f(t_m) = p_N(t_m)\,, \quad m = 0, 1, \ldots, N\,,$$

which gives

$$a_k^N = \frac{2}{N}\sum_{m=0}^{N}{}'' T_k^*(t_m) f(t_m) = a_k + \sum_{j=1}^{\infty} (a_{2Nj+k} + a_{2Nj-k})\,, \tag{23}$$

where we have used the Chebyshev series expansion (2) for $f(t)$ and the orthogonality of $T_k^*(t)$ on the abscissae t_m's in the second equality of (23). By making use of (23) in (2) and (3) it follows that

$$|f(t) - p_N(t)| \leq 2\sum_{k=N+1}^{\infty} |a_k|\,.$$

Assuming that $|a_k| = O(r^k)$ for a certain constant r, $0 \leq r \leq 1$, we have $\Sigma_{k=N+1}^{\infty}|a_k| =$

$|a_{N+1}|/(1-r)$. On the other hand, from (23) we have that $|a_N^N| \sim 2|a_N|$, provided N is sufficiently large and r is not near to 1. Consequently we may estimate the error of the approximation to (1) by e_N as follows

$$\left|\int_0^x e^{i\omega t}\{f(t)-p_N(t)\}\,dt\right| \leq \max_{0\leq t\leq 1} |f(t)-p_N(t)| \leq |a_N^N| r/(1-r) \equiv e_N . \tag{24}$$

In practice a constant r may be estimated from the asymptotic behaviour of the sequence $\{|a_k^N|\}$.

Next we consider the case that the approximate polynomial to $f(t)$ is $p_{3N/2}(t)$, where $3N/2 = 1.5 \times 2^n$. The authors [12] give the error bound for $p_{3N/2}(t)$ as follows

$$|f(t)-p_{3N/2}(t)| \leq 6 \sum_{k>3N/2} |a_k| .$$

Under the same assumption on the asymptotic behaviour of the Chebyshev coefficients $|a_k^{3N/2}|$ of $p_{3N/2}(t)$ as that on $|a_k^N|$ of $p_N(t)$ we have the error estimate

$$\left|\int_0^x e^{i\omega t}\{f(t)-p_{3N/2}(t)\}\,dt\right| \leq 6|a_{3N/2}^{3N/2}| r/(1-r) \equiv e_{3N/2} . \tag{25}$$

From (24) and (25) we observe that the constant factor 6 in the error estimate $e_{3N/2}$ for the approximate integral based on $p_{3N/2}(t)$ is six times greater than that in e_N for $p_N(t)$.

To evaluate the integral (1) to the required tolerance ε by using a polynomial $p_N(t)$ or $p_{3N/2}(t)$, $N = 2^n$, it is sufficient to determine the smallest degree N of the form $N = 2^n$, so that the estimate e_N or $e_{3N/2}$ given by (24) or (25), respectively, is no greater than ε. For an algorithm to generate recursively the sequence $p_8(t), p_{12}(t), \ldots, p_{2^n}(t), p_{1.5\times 2^n}(t), \ldots$, see Torii and Hasegawa [12].

4. Oscillatory infinite integral

We incorporate the integration scheme described in the preceeding sections into an automatic quadrature method for evaluating the infinite integral of oscillatory function I^c, (7). Let x_0 be the smallest zero of $\cos \omega x$ greater than a and determine $x_0 < x_1 < x_2 < \cdots$, where x_j is a root of $\cos \omega x$. Then we have

$$I^c = \sum_{j=0}^{\infty} I_j , \tag{26}$$

where

$$I_j = \int_{x_{j-1}}^{x_j} h(x) \cos \omega x \, dx , \quad 1 \leq j ,$$

$$I_0 = \int_a^{x_0} h(x) \cos \omega x \, dx . \tag{27}$$

If $h(x)$ is of constant sign and slowly decreases in magnitude to zero as $x \to \infty$, it is very difficult to evaluate I^c because the alternating series (26) is also of very slow convergence. Many acceleration methods for series have been applied to the series (26). An extrapolation method due to Sidi [9,10] seems to be most suitable for accelerating the series (26) provided the asymptotic behavior of the integrand is known.

Under an appropriate assumption on the asymptotic behavior of $h(x)$, say, $h(x) \sim x^{\gamma}$ ($\gamma < 0$) as $x \to \infty$, which is sufficient for the integral to converge, we have an asymptotic expansion for I^c

$$I^c \sim J(a, x_j) - \omega \sin(\omega x_j) h(x_j) \sum_{k=0}^{\infty} \beta_k / x_j^k , \tag{28}$$

where $J(a, x_j)$ denotes the integral on the finite interval $[a, x_j]$ defined by

$$J(a, x_j) = \sum_{k=0}^{j} I_k = \int_a^{x_j} h(x) \cos \omega x \, \mathrm{d}x , \tag{29}$$

and β_k's in (28) are constants. Applying a generalization of Richardson extrapolation method to the asymptotic expansion (28) we have more rapid convergent sequence (W transformation due to Sidi) than the original sequence $\{J(a, x_j)\}$. For details and algorithm of computing the W transformation recursively, see [8,9,10,11].

In addition to the use of the W transformation, for obtaining an efficient automatic quadrature scheme for I^c given by (7) it is necessary to evaluate the sequence $\{J(a, x_j)\}$, $j = 0, 1, 2, \ldots$, to the required tolerance ε with as small number of abscissae as possible. It has been usual to separately apply an appropriate quadrature rule to each of the definite integrals I_k's ($k = 0, 1, \ldots$) over half period $[x_{k-1}, x_k]$ and to add the resulting approximate values to obtain $J(a, x_j)$ given in (29).

Here we show that the method for evaluating the indefinite integral described in the preceding sections can be effectively used to compute several, say l, integrals $I_{s+1}, \ldots, I_{s+l}$ at a time for an arbitrary integer s. The number of abscissae actually needed depends on the way of the division of the interval $[a, x_j]$ of integration, that is, the choice of l, as well as on the character of $h(x)$. We propose an empirical way of dividing the interval. Let an integer l be the integral part of $0.7 \log_{10} \varepsilon^{-1} + 3$ and define intervals $K_0 = [a, x_2]$, $K_{q+1} = (x_{2+ql}, x_{2+(q+1)l}]$ ($q \geqslant 0$). We note that the integer l depends on the tolerance ε and so the interval K_{q+1} does. The results of the numerical experiments on the way of dividing the interval suggest that the choice of the integer l may be near optimal for the smooth and monotone decreasing function $h(x)$ in that the total number of abscissae required to achieve the requested accuracy ε for I^c is minimized. Let $J(K_q)$ denote the definite integral of $h(x) \cos \omega x$ on the interval K_q. If $x_j \in K_{\mu+1} = (x_{2+\mu l}, x_{2+(\mu+1)l}]$, then $[a, x_j] = (\bigcup_{q=0}^{\mu} K_q) \cup (x_{2+\mu l}, x_j]$. Consequently the integral (29) over $[a, x_j]$ is divided into the integrals over K_q ($0 \leqslant q \leqslant \mu$) and $(x_{2+\mu l}, x_j]$,

$$J(a, x_j) = \sum_{q=0}^{\mu} J(K_q) + J(x_{2+\mu l}, x_j) ,$$

where $J(x_{2+\mu l}, x_j)$ denotes the integral over the sub-interval $(x_{2+\mu l}, x_j]$ of $K_{\mu+1}$. These integrals $J(K_q)$ and $J(x_{2+\mu l}, x_j)$ are efficiently evaluated by using the indefinite integral (1), $I(\omega, x)$. Specifically, by the change of variables $x = Lu/\omega + x_{2+ql}$, the indefinite integral over K_{q+1} is given by

$$\int_{x_{2+ql}}^{x} h(t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \frac{L}{\omega}\exp[\mathrm{i}\omega x_{2+ql}]I(L, u)\,, \quad 0 \leqslant u \leqslant 1\,, \qquad x \in K_{q+1},\ 0 \leqslant q\,, \tag{30}$$

$$f(t) = h(Lt/\omega + x_{2+ql})\,, \quad 0 \leqslant t \leqslant 1\,, \tag{31}$$

where we have defined $L = \pi l$ for simplicity. It should be noted that the indefinite integral $I(L, u)$ depends on the interval K_{q+1}. The integral on the first interval K_0 must be separately treated,

$$\int_{a}^{x} h(t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = (2\pi/\omega + x_0 - a)\exp[\mathrm{i}\omega a]I(2\pi + \omega(x_0 - a), u)\,, \quad x \in K_0\,, \tag{32}$$

$$f(t) = h((2\pi/\omega + x_0 - a)t + a)\,. \tag{33}$$

By virtue of rapid convergence of the W transformation due to Sidi in practice only 3 intervals K_0, K_1 and K_2 at most, usually K_0, K_1 are sufficient to achieve the required accuracy ε for smooth and monotone decreasing functions $h(x)$.

If $h(x)$ is not a smooth function, for which the Chebyshev series is of slow convergence, we must choose an alternative value of l, which is difficult to be determined. Although reducing the value of l improves the rate of convergence of the Chebyshev series for $f(t)$ in (31), much more number of abscissae may be required to compute the total values of the input sequence to the W transformation, $J(a, x_j)$ $(j = 0, 1, \ldots)$, to the required tolerance ε. It is an open problem to determine an optimal value of l.

5. Numerical examples for infinite integral

We have computed the following integrals

(i) $\int_0^\infty \mathrm{e}^{-x}\cos x\,\mathrm{d}x = 0.5$,
(ii) $\int_0^\infty x/(x^2+1)\sin x\,\mathrm{d}x = \pi/(2\mathrm{e})$,
(iii) $\int_0^\infty 1/(x^2+1)\cos x\,\mathrm{d}x = \pi/(2\mathrm{e})$,
(iv) $\int_0^\infty 1/\sqrt{x^2+1}\cos x\,\mathrm{d}x = K_0(1)$,
(v) $\int_1^\infty 1/x^2\sin x\,\mathrm{d}x = \sin(1) - C_i(1)$,
(vi) $\int_0^\infty \log[(x^2+4)/(x^2+1)]\cos\omega x\,\mathrm{d}x = (\mathrm{e}^{-\omega} - \mathrm{e}^{-2\omega})\pi/\omega$.

In problem (iv) $K_0(x)$ is the modified Bessel function of the second kind. In (v) $C_i(x)$ is the cosine integral. Table 1 shows the numbers of abscissae required to satisfy the tolerance ε and the actual errors for the problems (i)–(v). For comparison the results by DQAWF of QUADPACK due to Piessens et al. [7] are also shown. In Table 2 the results for various values of ω of the problem (vi) are shown. In Table 2 we observe that the number of abscissae required to satisfy the absolute tolerance ε decreases as the value of ω increases because the exact value of the problem (vi) decreases with the exponential order $\mathrm{e}^{-\omega}$ as shown in the right hand side of the problem (vi). It can be seen from Tables 1 and 2 that the present scheme is effective for such the infinite integrals that the non-oscillatory part $h(x)$ of the integrand is smooth and has no peak except near the lower end of the interval $[a, \infty)$.

The computation was carried out in the double precision arithmetic (about 16 significant digits).

Table 1
Comparison of the performances of the present method and one due to Piessens et al. for oscillatory infinite integrals

Problem no.	$\varepsilon = 10^{-6}$ Present method		Piessens		$\varepsilon = 10^{-12}$ Present method		Piessens	
	N	Error	N	Error	N	Error	N	Error
(i)	33	3×10^{-9}	150	3×10^{-16}	61	5×10^{-16}	280	5×10^{-12}
(ii)	45	1×10^{-6}	385	3×10^{-9}	94	4×10^{-15}	700	8×10^{-13}
(iii)	57	1×10^{-8}	335	7×10^{-10}	93	1×10^{-13}	675	3×10^{-13}
(iv)	49	5×10^{-8}	360	2×10^{-9}	94	6×10^{-15}	720	1×10^{-12}
(v)	44	6×10^{-7}	330	8×10^{-10}	92	3×10^{-13}	670	7×10^{-13}

The numbers of abscissae required to satisfy the tolerance ε and the actual errors are given in the columns under the titles N and Error, respectively.

Table 2
Integral $\int_0^\infty \log[(x^2+4)/(x^2+1)] \cos \omega x \, dx$

	$\varepsilon = 10^{-6}$ Present method		Piessens		$\varepsilon = 10^{-12}$ Present method		Piessens	
ω	N	Error	N	Error	N	Error	N	Error
1	49	4×10^{-8}	335	2×10^{-9}	94	1×10^{-14}	670	7×10^{-13}
5	38	3×10^{-8}	300	1×10^{-9}	89	6×10^{-15}	680	8×10^{-13}
9	30	1×10^{-8}	275	3×10^{-9}	82	9×10^{-14}	765	3×10^{-13}
15	30	2×10^{-8}	275	1×10^{-9}	75	1×10^{-13}	725	8×10^{-16}

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